

# Construction of Directed Assortative Configuration Graphs

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## Abstract

Constructions of directed configuration graphs with given bi-degree distribution were introduced in random graph theory some years ago. These constructions lead to graphs where the degrees of two nodes belonging to the same edge are independent. However, it is observed that many real-life networks are assortative, meaning that edges tend to connect low degree nodes with high degree nodes, or variations thereof. In this article we provide an explicit algorithm to construct directed assortative configuration graphs with given bi-degree distribution and arbitrary pre-specified assortativity.

*Keywords:* random graphs; degree distribution; assortativity; assortative mixing; assortativity coefficient; degree correlations

## 1 Introduction

Random graphs are used to model large networks that consist of particles, called nodes, which are possibly linked to each other by edges. The study of random graphs goes back to the works of [8] and [9]. Since then, numerous random graph models have been introduced and studied in the literature. For an overview we refer the reader to [3, 6, 7, 17, 18]. Empirical studies of large data sets of real-life networks have shown that in many cases the degrees of two nodes belonging to the same edge are not independent (where the degree of a node is defined to be the number of edges attached to it). It is observed that in some types of real-life networks the degree of a node is positively related to the degrees of its linked neighbors, while in other situations the degree of a node is negatively related to the degrees of its linked neighbors. This property is called *assortativity* or *assortative mixing*. It has been discovered by [1, 5, 16] that financial networks typically show negative assortativity and that the strength of the assortativity influences the vulnerability of the financial network to shocks, see also [10]. In contrast, social networks tend to be positive assortative, see for instance [15]. More examples of assortative networks are presented in [14] and [12], where also quantities to measure assortativity in networks are proposed. On the other hand, there is only little literature about explicit constructions of random graphs showing

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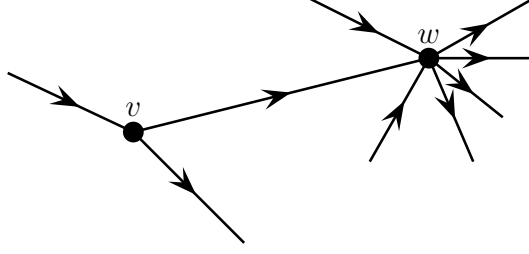


Figure 1: Node  $v$  is of type  $(1, 2)$  and node  $w$  is of type  $(3, 4)$ . Edge  $e = \langle v, w \rangle$  is of type  $(2, 3)$ .

assortative mixing. Established constructions of directed random graphs with given bi-degree distribution, called *configuration graphs*, lead to *non-assortative* graphs, see for instance the construction presented in [4]. Here, the bi-degree of a node  $v$  is a tuple  $(j_v, k_v)$ , where  $j_v$  is the number of edges arriving at node  $v$  (called *in-degree*) and  $k_v$  is the number of edges leaving from node  $v$  (called *out-degree*), and we say that node  $v$  is of type  $(j_v, k_v)$ , see Figure 1 for an illustration. In this article we extend the non-assortative construction presented in [4] by giving an explicit algorithm which allows to construct directed configuration graphs with a pre-specified assortativity based on a concept introduced in [11]. Namely, [11] proposed to specify the graph not only through their node-types, but also through their edge-types. We define the type of an edge  $e = \langle v, w \rangle$  connecting node  $v$  to node  $w$  by a tuple  $(k_e, j_e)$  with  $k_e$  denoting the out-degree of node  $v$  and  $j_e$  denoting the in-degree of node  $w$ , see Figure 1 for an illustration. This notion of edge-types is directly related to the notion of assortativity. In the positive assortative case  $k_e$  is positively related to  $j_e$  meaning that edges tend to connect nodes having similar degrees, and accordingly for the negative assortative case. If  $k_e$  is independent of  $j_e$ , then the graph is non-assortative. Therefore, [11] proposed to construct graphs with given node-type distribution  $P$  describing the nodes *and* with given edge-type distribution  $Q$  describing the edges, while different choices of  $Q$  result in different types of assortativity in the constructed graphs, see Figure 2 for examples. Nevertheless, there was not an explicit construction given in [11]. The aim of this article is to construct random graphs where nodes and edges follow pre-specified given bivariate distributions  $P$  and  $Q$ , and to give an explicit mathematical meaning to these distributions.

Let us first interpret the meaning of distributions  $P$  and  $Q$  in more detail. Node-type distribution  $P$  has the following interpretation. Assume we have a large directed network and we choose *at random* a node  $v$  of that network, then the type  $(j_v, k_v)$  of  $v$  has distribution  $P$ . Similarly, edge-type distribution  $Q$  should be understood as follows. If we choose *at random* an edge  $e$  of a large network, then its type  $(k_e, j_e)$  has distribution  $Q$ . This concept of  $P$  and  $Q$  distributions seems straightforward, however, it needs quite some care in order to give a rigorous mathematical meaning to these distributions, the difficulty lying in the “randomly” chosen node and edge obeying  $P$  and  $Q$ , respectively: the graph as total induces dependencies between nodes and

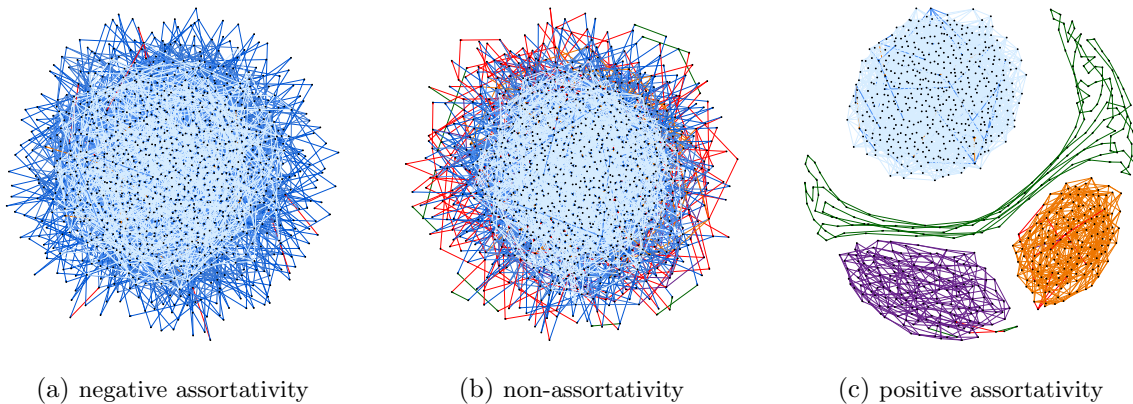


Figure 2: Three graphs generated by our algorithm given in Section 3 with  $N = 1000$  nodes. They all have the same node-type distribution  $P$  but different edge-type distributions  $Q$ . There are four different types of nodes present in each graph:  $(1, 1)$ ,  $(2, 2)$ ,  $(3, 3)$  and  $(4, 4)$ . Edges of identical type are colored the same. Edges that are arriving or leaving a highest-degree node are colored in different shades of blue. These edges are mainly present in the negative assortative case (a). In the positive assortative case (c), mainly nodes of the same type are connected. These edges are colored light blue, purple, orange and green. All other possible edges are colored red, which significantly appear only in (b).

edges which implies that the exact distributions can only be obtained in an asymptotic sense (this will be seen in the construction below).

We give an explicit algorithm to construct a directed assortative configuration graph with a given number of nodes and given distributions  $P$  and  $Q$ , and we prove that the type of a randomly chosen node of the resulting graph converges in distribution to  $P$  as the size of the graph tends to infinity. Similarly, the type of a randomly chosen edge converges in distribution to  $Q$ . These convergence results give a rigorous mathematical meaning to  $P$  and  $Q$  in line with their interpretation given above. The proposed algorithm allows for self-loops and multiple edges. In order to obtain a simple graph we delete all self-loops and multiple edges, and we show that the convergence results still hold true for the resulting simple graph. Recently, an alternative approach to construct assortative configuration graphs with given distributions  $P$  and  $Q$  was proposed in [10] using techniques from [20]. Our construction is different from [10] and more in the spirit of [2, 19, 4]. Moreover, we give a rigorous mathematical meaning to the given distributions  $P$  and  $Q$  which relies on the law of large numbers only.

In Section 2 we introduce the model and state our main results. Section 3 specifies the algorithm to generate directed assortative configuration graphs. The implementation of the algorithm in the programming language R can be downloaded from:

<https://people.math.ethz.ch/~wueth/Papers/AssortativeConfigurationGraphs.R>

In Section 4 we illustrate examples of assortative configuration graphs generated by our algorithm showing different assortative mixing. The proofs of the results are given in Section 5.

## 2 Model and main results

Consider fixed finite integers  $J \geq 1$  and  $K \geq 1$  which describe the maximal in- and out-degree of a node, respectively. For  $l \geq 0$  and  $n \geq l$  define  $[n]_l = \{l, \dots, n\}$ . For  $j \in [J]_0$  and  $k \in [K]_0$  we say that node  $v$  is of type  $(j, k)$  if the in-degree of  $v$  is  $j$  and the out-degree of  $v$  is  $k$ . For  $k \in [K]_1$  and  $j \in [J]_1$  we say that a directed edge  $e = \langle v, w \rangle$  is of type  $(k, j)$  if the out-degree of  $v$  is  $k$  and the in-degree of  $w$  is  $j$ . Figure 1 illustrates the notions of node- and edge-types. In the remainder, letter  $j$  always refers to in-degree and letter  $k$  to out-degree. Consider two bivariate probability distributions

$$\begin{aligned} P &= (p_{j,k})_{j \in [J]_0, k \in [K]_0} && \text{with } \sum_{j \in [J]_0, k \in [K]_0} p_{j,k} = 1; \\ Q &= (q_{k,j})_{k \in [K]_1, j \in [J]_1} && \text{with } \sum_{k \in [K]_1, j \in [J]_1} q_{k,j} = 1. \end{aligned}$$

We call  $P$  node-type distribution and  $Q$  edge-type distribution. We denote the respective marginal distributions of  $P$  and  $Q$  by

$$\begin{aligned} p_j^- &= \sum_{k' \in [K]_0} p_{j,k'} && \text{and } p_k^+ = \sum_{j' \in [J]_0} p_{j',k}, && j \in [J]_0 \text{ and } k \in [K]_0; \\ q_k^+ &= \sum_{j' \in [J]_1} q_{k,j'} && \text{and } q_j^- = \sum_{k' \in [K]_1} q_{k',j}, && k \in [K]_1 \text{ and } j \in [J]_1. \end{aligned}$$

In the remainder, superscript “ $-$ ” always refers to in-degree and superscript “ $+$ ” to out-degree. For instance,  $(p_j^-)_{j \in [J]_0}$  denotes the in-degree distribution of nodes. Observe that in a given graph the number of edges  $e = \langle v, w \rangle$  with out-degree of  $v$  being  $k \in [K]_1$  is equal to  $k$  times the number of nodes having out-degree  $k$ , and similarly for the number of nodes having in-degree  $j \in [J]_1$ . This relation between nodes and edges implies that we cannot choose  $P$  and  $Q$  independently of each other to achieve that nodes and edges in the constructed graph follow  $P$  and  $Q$ , respectively. We therefore assume that  $P$  and  $Q$  satisfy the following consistency conditions, see also [11] and [10], which implies that the above observation holds true in expectation in graphs where nodes and edges follow distributions  $P$  and  $Q$ , respectively.

$$q_k^+ = kp_k^+/z, \quad k \in [K]_1; \tag{C1}$$

$$q_j^- = jp_j^-/z, \quad j \in [J]_1, \tag{C2}$$

with *mean degree*  $z = \sum_{k \in [K]_0} kp_k^+$ . Observe that conditions (C1) and (C2) require  $z = \sum_{k \in [K]_0} kp_k^+ = \sum_{j \in [J]_0} jp_j^- > 0$ . This says that, in expectation, the sum of in-degrees equals the sum of out-degrees if nodes and edges follow distributions  $P$  and  $Q$ , respectively.

Given the number of nodes  $N \in \mathbb{N}$  and given distributions  $P$  and  $Q$  satisfying (C1) and (C2) the goal is to construct a graph such that the following statement is true in an asymptotic sense as the size  $N$  of the graph tends to infinity: the type of a randomly chosen node has distribution

$P$  and the type of a randomly chosen edge has distribution  $Q$ . The following theorem shows that this is indeed the case for graphs constructed by the algorithm provided in Section 3 and, hence, the theorem gives an explicit mathematical meaning to  $P$  and  $Q$ .

**Theorem 2.1** *Fix  $s \in \mathbb{N}$ . Let  $(j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s})$  be the types of  $s$  randomly chosen nodes of the graph generated by the algorithm provided in Section 3. Then,*

$$((j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s})) \xrightarrow{d} ((j'_1, k'_1), \dots, (j'_s, k'_s)), \quad \text{as } N \rightarrow \infty,$$

where  $(j'_1, k'_1), \dots, (j'_s, k'_s)$  are  $s$  independent random variables having distribution  $P$ . Similarly, the types of  $s$  randomly chosen edges converge in distribution, as  $N \rightarrow \infty$ , to a sequence of  $s$  independent random variables having distribution  $Q$ .

If we consider a graph where nodes and edges follow distributions  $P$  and  $Q$ , respectively, then we expect that the relative number of nodes of type  $(j, k)$  is close to  $p_{j,k}$  and that the relative number of edges of type  $(k, j)$  is close to  $q_{k,j}$ . Theorem 2.2 below makes this statement precise for graphs constructed by the algorithm provided in Section 3. To formulate the theorem, denote by  $\mathcal{V}_{j,k}$  the number nodes of type  $(j, k)$ ,  $j \in [J]_0$  and  $k \in [K]_0$ , and by  $\mathcal{E}_{k,j}$  the number edges of type  $(k, j)$ ,  $k \in [K]_1$  and  $j \in [J]_1$ , of the constructed graph of size  $N$ . The total number of edges we denote by  $\mathcal{E}$ . Theorem 2.2 says that the relative frequencies  $\mathcal{V}_{j,k}/N$  and  $\mathcal{E}_{k,j}/\mathcal{E}$  converge to  $p_{j,k}$  and  $q_{k,j}$ , respectively, in probability as  $N \rightarrow \infty$ .

**Theorem 2.2** *For the random graph constructed by the algorithm provided in Section 3 we have for any  $\varepsilon > 0$*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[ \sum_{j \in [J]_0, k \in [K]_0} \left| \frac{\mathcal{V}_{j,k}}{N} - p_{j,k} \right| + \sum_{k \in [K]_1, j \in [J]_1} \left| \frac{\mathcal{E}_{k,j}}{\mathcal{E}} - q_{k,j} \right| > \varepsilon \right] = 0.$$

The algorithm provided in Section 3 generates a graph possibly not being simple, i.e. it may contain self-loops and multiple edges. To obtain a simple graph we delete (erase) all self-loops and multiple edges, and we call the resulting graph *erased configuration graph*. The following theorem states that the asymptotic results still hold true for the erased configuration graph.

**Theorem 2.3** *The results of Theorem 2.1 and Theorem 2.2 still hold true for the erased configuration graph, based on the algorithm provided in Section 3.*

### 3 Construction of directed assortative configuration graphs

The algorithm to construct directed assortative configuration graphs starts from [4], where the authors construct a directed random graph with  $N \in \mathbb{N}$  nodes and given in-degree and given out-degree distributions in the following way. They assign to each node independently an in-degree and an out-degree according to the given distributions, also independently for different

nodes. Some degrees are then modified if the sum of in-degrees differs from the sum of out-degrees so that these sums of degrees are equal, and the sample is only accepted if the number of modifications is not too large. Finally, in-degrees are randomly paired with out-degrees. Note that this construction leads to a *non-assortative* configuration graph and the in- and out-degree of a given node are independent. The construction of an assortative configuration graph is more delicate since in-degrees cannot be randomly paired with out-degrees. In our construction we generate node-types using directly node-type distribution  $P$ . Independently of the node-types we generate  $zN$  edges having independent edge-types according to distribution  $Q$ . Finally, we match in- and out-degrees of nodes with edges of corresponding types. In general, the matching cannot be done exactly, but with high probability the number of types that need to be changed accordingly is small for large  $N$ , due to consistency conditions (C1) and (C2). We first describe the algorithm in detail and then comment on each step of the algorithm below.

**Algorithm to construct directed assortative configuration graphs.**

Assume maximal degrees  $J, K \geq 1$  and two probability distributions  $P$  and  $Q$  satisfying (C1) and (C2) with mean degree  $z$  are given. Choose  $\delta \in (1/2, 1)$  fixed. Choose  $N \in \mathbb{N}$  so large that there exists  $N' \in \mathbb{N}$  with  $N = N' + 2 \lceil N^\delta \rceil + \max\{J^2, K^2\}$ . Set  $N'' = N' + \lceil N^\delta \rceil$ .

**Step 1.** Assign to each node  $v = 1, \dots, N'$  independently a node-type  $(j_v, k_v)$  according to distribution  $P$ . Generate edges  $e = 1, \dots, \lceil zN'' \rceil$  having independent edge-types  $(k_e, j_e)$  according to distribution  $Q$ , independently of the node-types. Define

$$\begin{aligned} n_k^+ &= \sum_{v=1}^{N'} 1_{\{k_v=k\}} \quad \text{and} \quad e_k^+ = \left\lceil \frac{1}{k} \sum_{e=1}^{\lceil zN'' \rceil} 1_{\{k_e=k\}} \right\rceil \quad \text{for all } k \in [K]_1; \\ n_j^- &= \sum_{v=1}^{N'} 1_{\{j_v=j\}} \quad \text{and} \quad e_j^- = \left\lceil \frac{1}{j} \sum_{e=1}^{\lceil zN'' \rceil} 1_{\{j_e=j\}} \right\rceil \quad \text{for all } j \in [J]_1. \end{aligned}$$

Let  $A_N$  be the event on which we have

$$\begin{aligned} |n_k^+ - p_k^+ N'| &\leq p_k^+ N^\delta / 2 \quad \text{and} \quad |e_k^+ - p_k^+ N''| \leq p_k^+ N^\delta / 2 \quad \text{for all } k \in [K]_1; \\ |n_j^- - p_j^- N'| &\leq p_j^- N^\delta / 2 \quad \text{and} \quad |e_j^- - p_j^- N''| \leq p_j^- N^\delta / 2 \quad \text{for all } j \in [J]_1. \end{aligned}$$

Proceed to Step 2 if event  $A_N$  occurs. Otherwise, proceed to Step 5.

**Step 2.** For each  $k \in [K]_1$  and each  $j \in [J]_1$  do the following.

- Add  $r_k^+ = ke_k^+ - \sum_{e=1}^{\lceil zN'' \rceil} 1_{\{k_e=k\}}$  edges of type  $(k, 1)$ ;
- Add  $r_j^- = je_j^- - \sum_{e=1}^{\lceil zN'' \rceil} 1_{\{j_e=j\}}$  edges of type  $(1, j)$ .

Set  $r^+ = \sum_{k \in [K]_1} r_k^+$  and  $r^- = \sum_{j \in [J]_1} r_j^-$ .

**Step 3.** Set the type of each node in  $\{N' + 1, \dots, N\}$  to  $(0, 0)$ . For each  $k \in [K]_1$  and each  $j \in [J]_1$  do the following.

- Take the first  $e_k^+ - n_k^+$  nodes in  $\{N' + 1, \dots, N\}$  having out-degree 0 and change their out-degrees to  $k$ ;
- Take the first  $e_j^- - n_j^-$  nodes in  $\{N' + 1, \dots, N\}$  having in-degree 0 and change their in-degrees to  $j$ .

**Step 4.** For each  $k \in [K]_1$  and each  $j \in [J]_1$  do the following.

- Assign to each node having out-degree  $k$  exactly  $k$  uniformly chosen edges  $e$  of type  $(k_e, j_e)$  with  $k_e = k$ ;
- Assign to every node having in-degree  $j$  exactly  $j$  uniformly chosen edges  $e$  of type  $(k_e, j_e)$  with  $j_e = j$ .

Proceed to Step 6.

**Step 5.** Define node-types  $(j_1, k_1) = (0, 1)$ ,  $(j_2, k_2) = (1, 0)$  and  $(j_v, k_v) = (0, 0)$  for all  $v = 3, \dots, N$ . Insert an edge  $e = \langle 1, 2 \rangle$  that connects node 1 to node 2.

**Step 6.** Return the constructed graph.

### Explanation of the algorithm.

We say that node  $v$  is a  $k$ -node if its out-degree is  $k \in [K]_1$ , and similarly we say that edge  $e = \langle v, w \rangle$  is a  $k$ -edge if  $v$  is a  $k$ -node,  $k \in [K]_1$ .

**Step 1.** We generate only  $N'$  node-types and we keep  $2\lceil N^\delta \rceil + \max\{J^2, K^2\}$  nodes undetermined for possible modifications in later steps. The expected number of generated  $k$ -nodes is  $p_k^+ N'$  and the expected number of generated  $k$ -edges is  $q_k^+ \lceil zN'' \rceil$ . Using condition (C1), the expected number of  $k$ -nodes needed for the generated  $k$ -edges is therefore  $q_k^+ \lceil zN'' \rceil / k \approx p_k^+ N'' > p_k^+ N'$ . Henceforth, if  $e_k^+$  is close to its expectation  $q_k^+ \lceil zN'' \rceil / k \approx p_k^+ N''$ , it dominates the number of generated  $k$ -nodes  $n_k^+$  which is of order  $p_k^+ N' < p_k^+ N''$ . Step 3 is then used to correct for this imbalance in a deterministic way, and event  $A_N$  guarantees that this correction is possible. For receiving an efficient algorithm we would like event  $A_N$  to occur sufficiently likely, which is exactly stated in the next lemma.

**Lemma 3.1** *We have  $\mathbb{P}[A_N] \rightarrow 1$  as  $N \rightarrow \infty$ .*

Integer  $N''$  is chosen in such a way that, on event  $A_N$ , the number of  $k$ -nodes needed for the generated  $k$ -edges dominates the number of generated  $k$ -nodes, and their difference is at most  $2p_k^+ \lceil N^\delta \rceil$  for each  $k \in [K]_1$ , see also Figure 3 for an illustration. Therefore, on event  $A_N$ , the total number of additional nodes needed having a positive out-degree is

$$\sum_{k \in [K]_1} (e_k^+ - n_k^+) \leq 2\lceil N^\delta \rceil \sum_{k \in [K]_1} p_k^+ \leq 2\lceil N^\delta \rceil.$$

Hence, we have sufficiently many undetermined nodes in  $\{N' + 1, \dots, N\}$  to which we can assign out-degrees accordingly in Step 3, and similarly for the in-degrees.

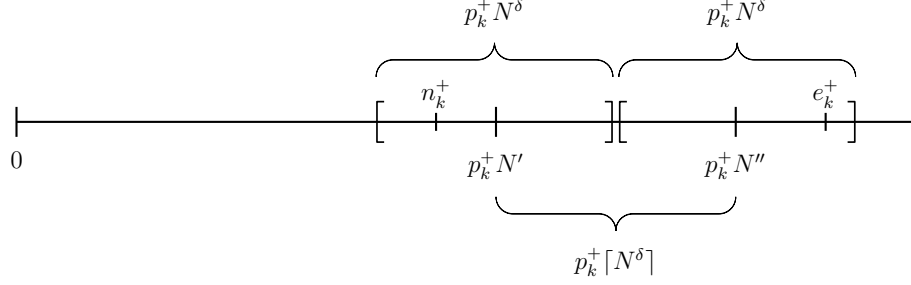


Figure 3: On event  $A_N$ , the number of generated  $k$ -nodes,  $n_k^+$ , lies in the interval of length  $p_k^+ N^\delta$  around  $p_k^+ N'$ . The number of  $k$ -nodes needed for the generated  $k$ -edges,  $e_k^+$ , lies in the interval of length  $p_k^+ N^\delta$  around  $p_k^+ N''$ . By definition of  $N''$  the gap between the two intervals is of size between 0 and  $p_k^+$ . From this it follows that there are at most  $2p_k^+ \lceil N^\delta \rceil$  additional  $k$ -nodes needed in order to attach all generated  $k$ -edges to  $k$ -nodes.

**Step 2.** In general, the number of generated  $k$ -edges is not a multiple of  $k$ . Therefore, we use Step 2 to correct for this cardinality by defining  $r_k^+$  additional edges of type  $(k, 1)$ . Note that each such edge requires a node having in-degree 1. Therefore, in total  $r^+ \leq K^2$  nodes having in-degree 1 are additionally needed, and similarly for the added edges of type  $(1, j)$ ,  $j \in [J]_1$ . The undetermined  $\max\{J^2, K^2\}$  nodes are exactly used to correct for the corresponding node-types in Step 3.

**Steps 3 and 4.** We assign node-types to the undetermined nodes  $N' + 1, \dots, N$  in such a way that the total number of  $k$ -nodes is equal to the number of  $k$ -nodes needed for the  $k$ -edges generated in Steps 1 and 2, for each  $k \in [K]_1$ , and similarly for the total number of nodes having in-degree  $j \in [J]_1$ . Then, all cardinalities for  $j$  and  $k$  match and all edges can be randomly connected to corresponding nodes. After doing so, each node has the correct number of arriving and leaving edges according to its type. Note that this step allows for self-loops and multiple edges.

**Step 5.** If event  $A_N$  does not occur in Step 2, we just define a deterministic graph so that all terms in Theorem 2.1 and Theorem 2.2 are well-defined. Due to Lemma 3.1 the influence of this deterministic graph is negligible.

## 4 Examples and discussion

Given a non-degenerate node-type distribution  $P$  with mean degree  $z > 0$  given by  $z = \sum_{k \in [K]_0} k p_k^+ = \sum_{j \in [J]_0} j p_j^-$ , we aim to find possible edge-type distributions  $Q$  such that  $P$  and  $Q$  satisfy (C1) and (C2). Conditions (C1) and (C2) imply that the marginal distributions of  $Q$  are fully described by the marginal distributions of  $P$ , and their respective cumulative distribution functions are given by

$$Q^+(k) = \sum_{k'=1}^k q_{k'}^+ = \frac{1}{z} \sum_{k'=1}^k k' p_{k'}^+ \quad \text{and} \quad Q^-(j) = \sum_{j'=1}^j q_{j'}^- = \frac{1}{z} \sum_{j'=1}^j j' p_{j'}^-,$$



for  $k \in [K]_1$  and  $j \in [J]_1$ . The possible joint distributions  $Q = (q_{k,j})_{k,j}$  are therefore given by

$$\begin{aligned} q_{k,j} = & C(Q^+(k), Q^-(j)) + C(Q^+(k-1), Q^-(j-1)) \\ & - C(Q^+(k), Q^-(j-1)) - C(Q^+(k-1), Q^-(j)), \end{aligned} \quad (4.1)$$

where  $C : [0, 1]^2 \rightarrow [0, 1]$  is a 2-dimensional copula, see for instance [13]. To measure assortativity of a graph, [14] introduced the *assortativity coefficient* of  $Q$  given by

$$\rho_Q = \frac{\sum_{k \in [K]_1} \sum_{j \in [J]_1} k j (q_{k,j} - q_k^+ q_j^-)}{\sqrt{\sum_{k \in [K]_1} k^2 q_k^+ - \left(\sum_{k \in [K]_1} k q_k^+\right)^2} \sqrt{\sum_{j \in [J]_1} j^2 q_j^- - \left(\sum_{j \in [J]_1} j q_j^-\right)^2}} \in [-1, 1],$$

which is Pearson's correlation coefficient of distribution  $Q$ . By Hoeffding's identity and using representation (4.1),  $\rho_Q$  can be rewritten as

$$\rho_Q = \frac{\sum_{k \in [K]_1} \sum_{j \in [J]_1} (C(Q^+(k), Q^-(j)) - Q^+(k)Q^-(j))}{\sqrt{\sum_{k \in [K]_1} k^2 q_k^+ - \left(\sum_{k \in [K]_1} k q_k^+\right)^2} \sqrt{\sum_{j \in [J]_1} j^2 q_j^- - \left(\sum_{j \in [J]_1} j q_j^-\right)^2}}.$$

Observe that  $\rho_Q$  is determined by  $P$  and  $C$ . Define the copulas

$$\begin{aligned} W(u_1, u_2) &= \max\{u_1 + u_2 - 1, 0\}; \\ M(u_1, u_2) &= \min\{u_1, u_2\}; \\ \Pi(u_1, u_2) &= u_1 u_2, \end{aligned}$$

for  $u_1, u_2 \in [0, 1]$ . Then,  $C = W$  corresponds to the minimal possible assortativity coefficient  $\rho_Q^- \in [-1, 0]$ , and  $C = M$  corresponds to the maximal possible assortativity coefficient  $\rho_Q^+ \in [0, 1]$ . Copula  $C = \Pi$  leads to non-assortativity and in this case we have  $q_{k,j} = k j p_k^+ p_j^- / z^2$  for all  $k \in [K]_1$  and  $j \in [J]_1$ . Note that for given  $P$ ,  $\rho_Q$  does not uniquely determine  $C$ . On the other hand, one can always find  $\lambda \in [0, 1]$  such that  $\lambda W + (1 - \lambda)M$  leads to a given assortativity coefficient  $\rho_Q \in [\rho_Q^-, \rho_Q^+]$ . This allows to construct directed assortative configuration graphs having any given assortativity coefficient that is possible for given node-type distribution  $P$ .

To illustrate assortativity in an example we consider maximal in- and out-degree  $J = K = 4$  and node-type distribution  $P_p = (p_{j,k}^p)$ ,  $p \in (0, 1)$ , given by

$$P_p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p \end{pmatrix}.$$

Distribution  $P_p$  only allows for nodes of types (2, 2) and (4, 4), with respective probabilities  $p$  and  $1 - p$ , which results in a mean degree of  $z = 4 - 2p$ . Clearly, these nodes can only be

connected through edges of types  $(2, 2)$ ,  $(2, 4)$ ,  $(4, 2)$  and  $(4, 4)$ . Since  $P_p$  is diagonal, consistency conditions (C1) and (C2) fully specify the edge-type distribution  $Q_q$  which is given by

$$Q_q = \frac{1}{2-p} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3p+q-2 & 0 & 2-2p-q \\ 0 & 0 & 0 & 0 \\ 0 & 2-2p-q & 0 & q \end{pmatrix},$$

for  $q = q(p) \in [\max\{2-3p, 0\}, 2-2p]$ . For fixed  $p \in (0, 1)$ , different values of  $q$  lead to different assortativity coefficients  $\rho_q = \rho_{Q_q}$ . A straightforward calculation gives

$$\rho_q = \frac{q(2-p) - 4(1-p)^2}{2p(1-p)}, \quad \text{or equivalently} \quad q = \frac{2(1-p)(2+p(\rho_q-2))}{2-p}.$$

For any  $p \in (0, 1)$ , the optimal bounds on  $\rho_q$  are given by

$$-\min \left\{ \frac{p}{2(1-p)}, \frac{2(1-p)}{p} \right\} = \rho_p^- \leq \rho_q \leq \rho_p^+ = 1.$$

Observe that  $\rho_p^- = -1$  if and only if  $p = 2/3$ . From now on we fix  $p = 0.5$ , meaning that there are nodes of types  $(2, 2)$  and  $(4, 4)$  with equal probability. For any  $q \in [0.5, 1]$  or  $\rho_q \in [-0.5, 1]$ , this leads to

$$\rho_q = 3q - 2 \in [-0.5, 1] \quad \text{or equivalently} \quad q = \frac{\rho_q + 2}{3} \in [0.5, 1].$$

A value of  $q = 1$  results in a graph with maximal assortativity coefficient  $\rho_1 = \rho_{0.5}^+ = 1$ . In this case, there are only edges that connect nodes having identical types. By decreasing  $q$  we allow also for edges of types  $(2, 4)$  and  $(4, 2)$ , while we reduce the probability of having edges of types  $(2, 2)$  and  $(4, 4)$ . If we decrease  $q$  to its minimal value  $q = 0.5$ , edges of type  $(2, 2)$  finally disappear and there are only edges of types  $(2, 4)$ ,  $(4, 2)$  and  $(4, 4)$ . This means that if  $q = 0.5$ , each edge is leaving from a node with maximal possible out-degree or is arriving at a node with maximal possible in-degree. In this case, the assortativity coefficient is negative and given by  $\rho_{0.5} = \rho_{0.5}^- = -0.5$ . Non-assortativity is given for  $q = 2/3$ . To illustrate these different types of assortativity, Figure 4 shows six graphs generated by the algorithm given in Section 3 with  $N = 1000$  nodes, with node-type distribution  $P_{0.5}$  and edge-type distribution  $Q_q$  for values of  $q$  such that  $\rho_q \in \{-0.5, 0, 0.5, 0.6, 0.8, 1\}$ . Note that the algorithm produces node-types different from  $(2, 2)$  and  $(4, 4)$  due to the modifications on nodes  $\{N' + 1, \dots, N\}$  with  $N - N' \ll N$  as  $N \rightarrow \infty$ . For better illustration we erase nodes of type  $(0, 0)$  and we erase self-loops and multiple edges. To illustrate the differences between the six generated graphs we color edges of identical types the same as follows:

- edges of type  $(2, 2)$  are orange,
- edges of types  $(2, 4)$  and  $(4, 2)$  are green,
- edges of type  $(4, 4)$  are blue,

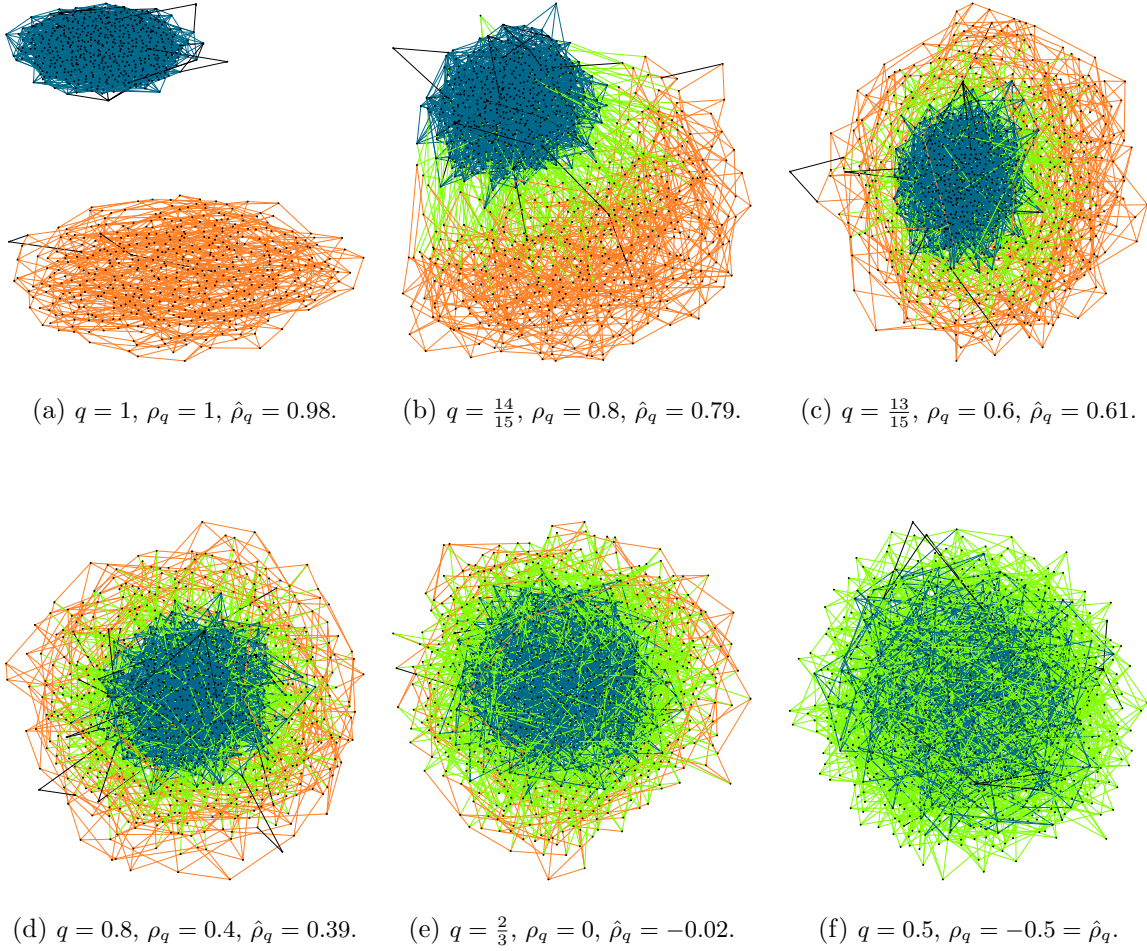


Figure 4: All graphs were generated by our algorithm in Section 3 with  $N = 1000$ , with the same node-type distribution  $P_{0.5} = \text{diag}(0, 0, 0.5, 0, 0.5)$  but different edge-type distribution  $Q_q$ . Edges of type  $(2, 2)$  are colored orange, while edges of type  $(4, 4)$  are colored blue. Edges of types  $(2, 4)$  and  $(4, 2)$  are colored green. All other edges are colored black.

and all other edges are colored black (which may arise by the construction and the erasure procedure). In Figure 4 we also present the values of the empirical assortativity coefficients  $\hat{\rho}_q$ . Note that they deviate from the actual assortativity coefficients because of the randomness in the construction and the erasure procedure. Nevertheless, by Theorem 2.3 and by the continuous mapping theorem, the empirical assortativity coefficient  $\hat{\rho}_Q$  converges in probability to  $\rho_Q$  as  $N \rightarrow \infty$ .

## 5 Proofs

We start with the proof of Lemma 3.1 which states that  $\mathbb{P}[A_N] \rightarrow 1$  as  $N \rightarrow \infty$ .

**Proof of Lemma 3.1.** For each  $k \in [K]_1$ ,  $n_k^+$  has a binomial distribution with parameters  $N'$  and  $p_k^+$ . Therefore,

by Chebyshev's inequality,

$$\mathbb{P} \left[ |n_k^+ - p_k^+ N'| > p_k^+ N^\delta / 2 \right] \leq \frac{N' p_k^+ (1 - p_k^+)}{N^{2\delta} (p_k^+)^2 / 4} \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

since  $\delta \in (1/2, 1)$ . Similarly,  $\sum_{e=1}^{\lceil zN'' \rceil} 1_{\{k_e=k\}}$  has a binomial distribution with parameters  $\lceil zN'' \rceil$  and  $q_k^+$ . Therefore, by condition (C1),  $\sum_{e=1}^{\lceil zN'' \rceil} 1_{\{k_e=k\}}/k$  has mean  $q_k^+ \lceil zN'' \rceil / k = p_k^+ \lceil zN'' \rceil / z$  and the variance is of order  $N''$ . By Chebyshev's inequality it follows that

$$\mathbb{P} \left[ |e_k^+ - p_k^+ N''| > p_k^+ N^\delta / 2 \right] \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Similarly for  $n_j^-$  and  $e_j^-$ ,  $j \in [J]_1$ , using condition (C2).  $\square$

**Proof of Theorem 2.1.** Choose  $s \in \mathbb{N}$  fixed and let  $u : (\mathbb{N}_0 \times \mathbb{N}_0)^s \rightarrow [-H, H]$  be a continuous function bounded by  $H > 0$ . For  $N \in \mathbb{N}$ , denote by  $(j_1, k_1), \dots, (j_N, k_N)$  the node-types generated by the algorithm in Section 3. Define the coupling  $((\tilde{j}_1, \tilde{k}_1), \dots, (\tilde{j}_N, \tilde{k}_N))$ , where  $(\tilde{j}_v, \tilde{k}_v) = (j_v, k_v)$  for all  $v = 1, \dots, N'$ , and  $(\tilde{j}_v, \tilde{k}_v)$ ,  $v = N' + 1, \dots, N$ , are independent random variables each having distribution  $P$ . By the triangle inequality we have

$$\begin{aligned} & \left| \mathbb{E} [u((j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s}))] - \mathbb{E} [u((j'_1, k'_1), \dots, (j'_s, k'_s))] \right| \\ & \leq \left| \mathbb{E} [u((j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s})) - u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s}))] \right| \\ & \quad + \left| \mathbb{E} [u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s})) - u((j'_1, k'_1), \dots, (j'_s, k'_s))] \right|. \end{aligned}$$

Since on  $A_N$ ,  $((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s}))$  has the same distribution as  $((j'_1, k'_1), \dots, (j'_s, k'_s))$ , the second term on the right-hand side satisfies

$$\begin{aligned} & \left| \mathbb{E} [u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s})) - u((j'_1, k'_1), \dots, (j'_s, k'_s))] \right| \\ & \leq \left| \mathbb{E} [u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s})) - u((j'_1, k'_1), \dots, (j'_s, k'_s))] \middle| A_N \right| + 2H \mathbb{P}[A_N^c] = 2H \mathbb{P}[A_N^c], \end{aligned}$$

which converges to 0 as  $N \rightarrow \infty$  by Lemma 3.1. For the first term we have by the definition of the coupling

$$\begin{aligned} & \left| \mathbb{E} [u((j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s})) - u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s}))] \right| \\ & = \left| \mathbb{E} \left[ \left( u((j_{v_1}, k_{v_1}), \dots, (j_{v_s}, k_{v_s})) - u((\tilde{j}_{v_1}, \tilde{k}_{v_1}), \dots, (\tilde{j}_{v_s}, \tilde{k}_{v_s})) \right) 1_{\bigcup_{l=1}^s \{v_l \in \{N'+1, \dots, N\}\}} \right] \right| \\ & \leq 2H \sum_{l=1}^s \mathbb{P}[v_l \in \{N'+1, \dots, N\}] = 2Hs \frac{N - N'}{N}, \end{aligned}$$

which converges to 0 as  $N \rightarrow \infty$  by the choice of  $N'$ . The corresponding result for the edge-types follows by exactly the same arguments since, on event  $A_N$ , the number of generated edge-types having distribution  $Q$  is  $\lceil zN'' \rceil$  and the number of artificially added edge-types is at most  $K^2 + J^2$ , see Step 2 of the algorithm.  $\square$

**Proof of Theorem 2.2.** Let  $j \in [J]_0$ ,  $k \in [K]_0$  and choose  $\varepsilon > 0$ . For  $N$  so large that  $(N - N')/N \leq \varepsilon/2$  we have

$$\begin{aligned} \mathbb{P} \left[ \left| \frac{1}{N} \sum_{v=1}^N 1_{\{j_v=j, k_v=k\}} - p_{j,k} \right| > \varepsilon \right] & \leq \mathbb{P} \left[ \left| \frac{N - N'}{N} + \frac{1}{N} \sum_{v=1}^{N'} 1_{\{j_v=j, k_v=k\}} - p_{j,k} \right| > \varepsilon \middle| A_N \right] + \mathbb{P}[A_N^c] \\ & \leq \mathbb{P} \left[ \left| \sum_{v=1}^{N'} 1_{\{j_v=j, k_v=k\}} - p_{j,k} N \right| > N\varepsilon/2 \middle| A_N \right] + \mathbb{P}[A_N^c]. \end{aligned}$$

By Lemma 3.1 it remains to consider the first term on the right-hand side. By the triangle and Chebyshev's inequality it follows that

$$\mathbb{P} \left[ \left| \sum_{v=1}^{N'} 1_{\{j_v=j, k_v=k\}} - p_{j,k} N \right| > N\varepsilon/2 \middle| A_N \right] \leq \frac{N' p_{j,k} (1 - p_{j,k})}{\mathbb{P}[A_N] N^2 \varepsilon^2 / 16} + \mathbb{P}[(N - N') p_{j,k} > N\varepsilon/4 \middle| A_N],$$

which converges to 0 as  $N \rightarrow \infty$  by Lemma 3.1. Similarly for the edge-types.  $\square$

In order to prove Theorem 2.3, we first show that the expected number of self-loops and multiple edges arising from the construction in Section 3 is bounded in  $N$ .

**Lemma 5.1** *Let  $S_N$  be the number of self-loops and  $M_N$  be the number of multiple edges of the graph generated by the algorithm in Section 3. There exists a finite constant  $C > 0$  such that*

$$\mathbb{E}[S_N + M_N | A_N] \leq C.$$

**Proof of Lemma 5.1.** Let  $v \in \{1, \dots, N\}$  and denote by  $s_v$  the number of edges  $e$  with  $e = \langle v, v \rangle$ . The probability that an edge leaving from node  $v$  is also arriving at node  $v$  is at most 1 divided by the number of nodes having in-degree  $j_v$ , which we denote by  $\mathcal{V}_{j_v}^- \geq 1$ . Therefore, since there are  $k_v$  edges leaving from  $v$ ,

$$\mathbb{E}[S_N | A_N] = \sum_{v=1}^N \mathbb{E}[s_v | A_N] \leq \sum_{v=1}^N \mathbb{E}\left[\frac{k_v}{\mathcal{V}_{j_v}^-} \middle| A_N\right].$$

On event  $A_N$ , the number of nodes having in-degree  $j_v$  is at least  $\max\{1, p_{j_v}^- N' - p_{j_v}^- N^\delta / 2\}$ . It follows that

$$\mathbb{E}[S_N | A_N] \leq \sum_{v=1}^N \mathbb{E}\left[\frac{k_v}{\max\{1, p_{j_v}^- N' - p_{j_v}^- N^\delta / 2\}} \middle| A_N\right] \leq \frac{NK}{\max\{1, \min_{j \in [J]_1} \{p_j^- N' - p_j^- N^\delta / 2\}\}}.$$

Since  $z > 0$ , there exists  $j \in [J]_1$  with  $p_j^- > 0$  and, hence, the right-hand side is bounded in  $N$ .

To bound the expectation of  $M_N$ , let  $v \in \{1, \dots, N\}$  and denote by  $m_v$  the number of multiple edges leaving from node  $v$ . The probability that two distinct edges leaving from  $v$  are arriving at the same node  $w \neq v$  is at most

$$\frac{j_w(j_w - 1)}{j_w \mathcal{V}_{j_w}^- (j_w \mathcal{V}_{j_w}^- - 1)} \mathbf{1}_{\{j_w \geq 2\}}.$$

It follows that

$$\begin{aligned} \mathbb{E}[M_N | A_N] &= \sum_{v=1}^N \mathbb{E}[m_v | A_N] \leq \sum_{v=1}^N \sum_{w=1}^N \mathbb{E}\left[\binom{k_v}{2} \frac{j_w(j_w - 1)}{j_w \mathcal{V}_{j_w}^- (j_w \mathcal{V}_{j_w}^- - 1)} \mathbf{1}_{\{j_w \geq 2\}} \mathbf{1}_{\{k_v \geq 2\}} \middle| A_N\right] \\ &\leq \sum_{v=1}^N \sum_{w=1}^N \mathbb{E}\left[\frac{k_v^2 j_w^2}{2 j_w \mathcal{V}_{j_w}^- (j_w \mathcal{V}_{j_w}^- - 1)} \mathbf{1}_{\{j_w \geq 2\}} \mathbf{1}_{\{k_v \geq 2\}} \middle| A_N\right]. \end{aligned}$$

Since  $2 j_w \mathcal{V}_{j_w}^- (j_w \mathcal{V}_{j_w}^- - 1) \geq (j_w \mathcal{V}_{j_w}^-)^2$  for  $j_w \geq 2$ , it follows that

$$\mathbb{E}[M_N | A_N] \leq \sum_{v=1}^N \sum_{w=1}^N \mathbb{E}\left[\frac{k_v^2}{(\mathcal{V}_{j_w}^-)^2} \mathbf{1}_{\{j_w \geq 2\}} \mathbf{1}_{\{k_v \geq 2\}} \middle| A_N\right].$$

Using that on  $A_N$ , the number of nodes having in-degree  $j_w$  is at least  $\max\{1, p_{j_w}^- N' - p_{j_w}^- N^\delta / 2\}$ , it follows that

$$\mathbb{E}[M_N | A_N] \leq \frac{N^2 K^2}{\max\{1, \min_{j \in [J]_1} \{p_j^- N' - p_j^- N^\delta / 2\}\}^2}.$$

The right-hand side is again bounded in  $N$ . This finishes the proof of Lemma 5.1.  $\square$

**Proof of Theorem 2.3.** We first show that Theorem 2.2 holds true for the erased configuration graph. Let  $S_N$  be the number of self-loops and let  $M_N$  be the number of multiple edges generated by the algorithm. For

$j \in [J]_0$  and  $k \in [K]_0$  denote by  $\mathcal{V}_{j,k}^e$  the number of constructed nodes of type  $(j, k)$  after erasing all self-loops and multiple edges. Similarly we define  $\mathcal{E}_{k,j}^e$  for  $k \in [K]_1$  and  $j \in [J]_1$ . In order to prove that Theorem 2.2 holds true for the erased configuration graph, we show that for any  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[ \sum_{j \in [J]_0, k \in [K]_0} \left| \frac{\mathcal{V}_{j,k}^e}{N} - p_{j,k} \right| \geq \varepsilon \quad \text{or} \quad \sum_{k \in [K]_1, j \in [J]_1} |\mathcal{E}_{k,j}^e - q_{k,j} \mathcal{E}^e| \geq \varepsilon \mathcal{E}^e \right] = 0, \quad (5.1)$$

where  $\mathcal{E}^e$  denotes the total number of edges in the erased configuration graph (which could be equal to 0 if all edges of the constructed graph are self-loops). Let  $\varepsilon > 0$  and choose  $j \in [J]_0$  and  $k \in [K]_0$ . We have

$$\mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}^e}{N} - p_{j,k} \right| \geq \varepsilon \right] \leq \mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}^e - \mathcal{V}_{j,k}}{N} \right| \geq \varepsilon/2 \right] + \mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}}{N} - p_{j,k} \right| \geq \varepsilon/2 \right].$$

The second term on the right-hand side converges to 0 as  $N \rightarrow \infty$  by Theorem 2.2. For the first term, note that

$$\mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}^e - \mathcal{V}_{j,k}}{N} \right| \geq \varepsilon/2 \right] \leq \mathbb{P} \left[ \sum_{v=1}^N |1_{\{j_v^e=j, k_v^e=k\}} - 1_{\{j_v=j, k_v=k\}}| \geq \varepsilon N/2 \right],$$

where  $(j_v^e, k_v^e)$  denotes the type of node  $v$  in the erased configuration graph. Denote by  $s_v$  the number of self-loops attached to  $v$ . Denote by  $m_v^+$  the number of multiple edges leaving from  $v$  and denote by  $m_v^-$  the number of multiple edges arriving at  $v$ . Note that  $j_v^e = j_v$  if and only if  $s_v + m_v^- = 0$ , and similarly  $k_v^e = k_v$  if and only if  $s_v + m_v^+ = 0$ . We therefore have that

$$\begin{aligned} 1_{\{j_v^e=j, k_v^e=k\}} - 1_{\{j_v=j, k_v=k\}} &= 1_{\{s_v+m_v^->0 \text{ or } s_v+m_v^+>0\}} (1_{\{j_v^e=j, k_v^e=k\}} - 1_{\{j_v=j, k_v=k\}}) \\ &= 1_{\{s_v+m_v>0\}} (1_{\{j_v^e=j, k_v^e=k\}} - 1_{\{j_v=j, k_v=k\}}), \end{aligned}$$

where we set  $m_v = m_v^+ + m_v^-$ . Hence,

$$\mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}^e - \mathcal{V}_{j,k}}{N} \right| \geq \varepsilon/2 \right] \leq \mathbb{P} \left[ \sum_{v=1}^N 1_{\{s_v+m_v>0\}} \geq \varepsilon N/2 \right] \leq \mathbb{P} \left[ \sum_{v=1}^N (s_v + m_v) \geq \varepsilon N/2 \right].$$

It follows by Markov's inequality, Lemma 3.1 and Lemma 5.1 that

$$\mathbb{P} \left[ \left| \frac{\mathcal{V}_{j,k}^e - \mathcal{V}_{j,k}}{N} \right| \geq \varepsilon/2 \right] \leq \mathbb{P} [S_N + 2M_N \geq N\varepsilon/2] \leq \frac{\mathbb{E}[S_N + 2M_N | A_N]}{N\varepsilon/2} + \mathbb{P}[A_N^c] \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

We now prove the same result for the edge-types under the conditional probability, conditional given  $A_N$ , which is enough due to Lemma 3.1. Note that on event  $A_N$  the number of generated edges  $\mathcal{E}$  is at least  $zN''$ , see Step 1 of the algorithm. It follows by Markov's inequality and Lemma 5.1, and since  $\mathcal{E} - \mathcal{E}^e = S_N + M_N$ , that for every  $\eta > 0$ ,

$$\mathbb{P} \left[ \left| \frac{\mathcal{E}^e}{\mathcal{E}} - 1 \right| \geq \eta \mid A_N \right] = \mathbb{P} \left[ \frac{S_N + M_N}{\mathcal{E}} \geq \eta \mid A_N \right] \leq \mathbb{P} [S_N + M_N \geq zN''\eta \mid A_N] \rightarrow 0, \quad (5.2)$$

as  $N \rightarrow \infty$ . Note that for fixed  $k \in [K]_1$  and  $j \in [J]_1$ ,

$$\mathbb{P} [|\mathcal{E}_{k,j}^e - q_{k,j} \mathcal{E}^e| \geq \varepsilon \mathcal{E}^e \mid A_N] \leq \mathbb{P} [|\mathcal{E}_{k,j}^e - \mathcal{E}_{k,j}| \geq \varepsilon \mathcal{E}^e/2 \mid A_N] + \mathbb{P} [|\mathcal{E}_{k,j} - q_{k,j} \mathcal{E}^e| \geq \varepsilon \mathcal{E}^e/2 \mid A_N].$$

For the second term on the right-hand side we have

$$\begin{aligned} \mathbb{P} [|\mathcal{E}_{k,j} - q_{k,j} \mathcal{E}^e| \geq \varepsilon \mathcal{E}^e/2 \mid A_N] &= \mathbb{P} \left[ \left| \frac{\mathcal{E}_{k,j}}{\mathcal{E}} - \frac{\mathcal{E} - S_N - M_N}{\mathcal{E}} q_{k,j} \right| \geq \frac{\mathcal{E}^e}{\mathcal{E}} \varepsilon/2 \mid A_N \right] \\ &\leq \mathbb{P} \left[ \left| \frac{\mathcal{E}_{k,j}}{\mathcal{E}} - q_{k,j} \right| \geq \frac{\mathcal{E}^e}{\mathcal{E}} \varepsilon/4 \mid A_N \right] + \mathbb{P} [(S_N + M_N) q_{k,j} \geq \mathcal{E}^e \varepsilon/4 \mid A_N] \\ &\leq \mathbb{P} \left[ \left| \frac{\mathcal{E}_{k,j}}{\mathcal{E}} - q_{k,j} \right| \geq \frac{\mathcal{E}^e}{\mathcal{E}} \varepsilon/4 \mid A_N \right] + \mathbb{P} \left[ S_N + M_N \geq \frac{zN''\varepsilon/4}{q_{k,j} + \varepsilon/4} \mid A_N \right], \end{aligned}$$

where in the last step we used that  $\mathcal{E}^e = \mathcal{E} - (S_N + M_N) \geq zN'' - (S_N + M_N)$ . By Theorem 2.2, (5.2) and Lemma 5.1, the right-hand side converges to 0 as  $N \rightarrow \infty$ . To bound the first term, note that the erasure

procedure changes the number of edges of type  $(k, j)$  because such edges may get erased, but also because an erased edge  $e = \langle v, w \rangle$  changes the types of unerased edges that are leaving from node  $v$  or that are arriving at node  $w$ . It follows that

$$\mathbb{P} \left[ |\mathcal{E}_{k,j}^e - \mathcal{E}_{k,j}| \geq \varepsilon \mathcal{E}^e / 2 \mid A_N \right] \leq \mathbb{P} \left[ \left| S_N + M_N + \sum_{e=1}^{\mathcal{E}^e} |1_{\{k_e^e=k, j_e^e=j\}} - 1_{\{k_e=k, j_e=j\}}| \right| \geq \varepsilon \mathcal{E}^e / 2 \mid A_N \right],$$

where  $(k_e^e, j_e^e)$  denotes the type of edge  $e$  in the erased configuration graph. For an edge  $e = \langle v, w \rangle$  we have  $k_e^e = k_e$  if and only if  $s_v + m_v^+ = 0$ , and similarly  $j_e^e = j_e$  if and only if  $s_w + m_w^- = 0$ . Therefore, for  $e = \langle v, w \rangle$ ,

$$\begin{aligned} 1_{\{k_e^e=k, j_e^e=j\}} - 1_{\{k_e=k, j_e=j\}} &= 1_{\{s_v+m_v^+>0 \text{ or } s_w+m_w^->0\}} (1_{\{k_e^e=k, j_e^e=j\}} - 1_{\{k_e=k, j_e=j\}}) \\ &= 1_{\{s_e+m_e>0\}} (1_{\{k_e^e=k, j_e^e=j\}} - 1_{\{k_e=k, j_e=j\}}), \end{aligned}$$

where we set  $s_e = s_v + s_w$  and  $m_e = m_v^+ + m_w^-$ . It follows that

$$\begin{aligned} \mathbb{P} \left[ |\mathcal{E}_{k,j}^e - \mathcal{E}_{k,j}| \geq \varepsilon \mathcal{E}^e / 2 \mid A_N \right] &\leq \mathbb{P} \left[ \left| S_N + M_N + \sum_{e=1}^{\mathcal{E}^e} 1_{\{s_e+m_e>0\}} \right| \geq \varepsilon \mathcal{E}^e / 2 \mid A_N \right] \\ &\leq \mathbb{P} \left[ \left| S_N + M_N + \sum_{v=1}^N (k_v 1_{\{s_v+m_v^+>0\}} + j_v 1_{\{s_v+m_v^->0\}}) \right| \geq \varepsilon \mathcal{E}^e / 2 \mid A_N \right] \\ &\leq \mathbb{P} [S_N + M_N + K(S_N + M_N) + J(S_N + M_N) \geq \varepsilon \mathcal{E}^e / 2 \mid A_N], \end{aligned}$$

which converges to 0 as  $N \rightarrow \infty$  by Lemma 5.1 and the fact that  $\mathcal{E}^e = \mathcal{E} - (S_N + M_N) \geq zN'' - (S_N + M_N)$ . This finally proves (5.1).

We now prove that Theorem 2.1 holds true for the erased configuration graph. Denote by  $\mathcal{V}_{\text{mod}}$  the set of nodes whose types have been changed due to the erasure procedure. For the probability that a uniformly chosen node  $v$  belongs to  $\{N' + 1, \dots, N\} \cup \mathcal{V}_{\text{mod}}$  we have

$$\mathbb{P} [v \in \{N' + 1, \dots, N\} \cup \mathcal{V}_{\text{mod}}] = \mathbb{E} \left[ \frac{|\{N' + 1, \dots, N\} \cup \mathcal{V}_{\text{mod}}|}{N} \right] \leq \mathbb{E} \left[ \frac{N - N' + S_N + 2M_N}{N} \right].$$

Since the graph returned by the algorithm in case event  $A_N$  does not hold has no self-loops or multiple edges, see Step 5, it follows that

$$\mathbb{P} [v \in \{N' + 1, \dots, N\} \cup \mathcal{V}_{\text{mod}}] \leq \mathbb{E} \left[ \frac{N - N' + S_N + 2M_N}{N} \mid A_N \right] + \frac{N - N'}{N},$$

which converges to 0 as  $N \rightarrow \infty$  by Lemma 5.1. Going through the proof of Theorem 2.1, we see that this observation is enough to conclude that the types of  $s \in \mathbb{N}$  randomly chosen nodes of the erased configuration graph converge in distribution to a sequence of  $s$  independent random variables each having distribution  $P$  as  $N \rightarrow \infty$ . To prove the same result for the edge-types of the erased configuration graph, note that it may happen that all edges of the graph generated by the algorithm of Section 3 are self-loops, i.e.  $\mathcal{E}^e = 0$ . In this case the edge set is empty for the erased configuration graph and we define “the type of a randomly chosen edge” to be identical to  $(1, 1)$  if event  $\{\mathcal{E}^e = 0\}$  occurs, and we define it to be as usual if event  $\{\mathcal{E}^e = 0\}$  does not occur. Nevertheless, the probability of event  $\{\mathcal{E}^e = 0\}$  converges to 0 as  $N \rightarrow \infty$  by Lemma 3.1 and (5.2) above. Therefore, using similar arguments as above for the node-types, we conclude that the types of  $s \in \mathbb{N}$  randomly chosen edges of the erased configuration graph converge in distribution to a sequence of  $s$  independent random variables each having distribution  $Q$  as  $N \rightarrow \infty$ .  $\square$

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